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A note on the flow in a trailing vortex

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SUMMARY

We show that if the equations governing the fluid motion in a trailing vortex is linearized as by Batchelor, more than one solution can be constructed. Within the framework of the linear theory, there is no criterion to determine which solution is to be used. To clarify the situation, we formulate the Navier–Stokes equations in parabolic coordinates and seek asymptotic solutions valid far downstream. By insisting that the interaction of the swirl with the uniform stream be a first order effect, we obtain the first two terms in the asymptotic expansions for the Stokes stream function and the angular momentum. The result thus obtained differs from that given by Batchelor in that the axial velocity defect decays algebraically.

1. Introduction

Studies of the fluid motion in a trailing vortex has a large literature. In 1959, Newman [7] considered a fairly simple model in which the governing equations are linearized and decoupled. In 1964 Batchelor [1] considered a coupled, linearized system and obtained a solution qualitatively different from that of Newman. The survey article by Hall [5] contains extensive references to theoretical and experimental investigations. The original purpose of this investigation was to extend the linearized solution obtained by Batchelor to an asymptotic expansion for the non-linear system. However, it was soon realized that the linear system in cylindrical polar coordinates used by Batchelor admits two solutions. To clarify the situation, we formulate the Navier–Stokes equation in parabolic coordinates, a choice made obvious by the linear solution, and then obtain an asymptotic expansion far downstream. In this way, we arrive at a solution which resembles one of the solutions found for the linear system, and it is different from the one obtained by Batchelor. As usual, the higher order solutions in the expansion contain constants not determinable by the boundary conditions alone. Additional information on these constants is obtained by using integral relations derived from the momentum equations.

In section 2 we re-examine the linear system and exhibit the two solutions. In section 3 we formulate the problem in parabolic coordinates, and construct an asymptotic expansion.

2. The linear system

Let (x, r, ϕ) denote cylindrical coordinates with corresponding velocity components (u, v, w). Let p denote the pressure and $C = rw$. If we consider the steady, rotationally symmetric motion of a viscous incompressible fluid, and assume that $\partial/\partial x \ll \partial/\partial r$; $v \ll u$, and $|u-U| \ll U$; then the Navier–Stokes equations can be approximated by [1]

$$U\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) \tag{1}$$

$$U\frac{\partial C}{\partial x} = \nu r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial C}{\partial r}\right) \tag{2}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = \frac{C^2}{r^3}. \tag{3}$$

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The boundary conditions we impose on $C(x, r)$ and $u(x, r)$ are

$$C(x, 0) = 0 \; ; \quad C(x, \infty) = \Gamma \; ; \quad u(x, \infty) = U \, . \tag{4}$$

No condition is imposed on $u(x, 0)$, but we require that it be capable of accounting for a velocity defect.

In 1959, Newman [7] considered a similar system of equations, except that the term $-\rho^{-1}\partial p/\partial x$ in (1) is absent. Thus, u and C are uncoupled.

Equations (1), (2) and (3) were solved by Batchelor, who obtained

$$C = \Gamma(1 - e^{-\eta}) \tag{5}$$

$$u = U - \frac{\Gamma^2}{8\nu x}\left[\log \frac{xU}{\nu} Q_1(\eta) - Q_2(\eta)\right] - \frac{LU^2}{8\nu x} e^{-\eta} \, , \tag{6}$$

where $\eta = Ur^2/4\nu x$, $Q_1(\eta) = e^{-\eta}$, $Q_2(\eta) = e^{-\eta}\{\log \eta + \mathrm{ei}(\eta) - 0.807\} - 2\mathrm{ei}(2\eta) + 2\mathrm{ei}(\eta)$.

In (6), L is a constant with the dimension of area and the last term accounts for any initial velocity defect which may be independent of the circulation. The function $\mathrm{ei}(\eta) = \int_\eta^\infty t^{-1} e^{-t} dt$ is tabulated.

Since the solution to be constructed for the non-linear system is based on the linearized solution we would like to re-examine the asymptotic form for u. Now (2) and (3) can be solved to give p in terms of η and (1) then becomes [1]

$$U \frac{\partial u}{\partial x} - \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) = -\frac{\Gamma^2 U}{8\nu x}\left(P + \eta \frac{dP}{d\eta}\right) \tag{7}$$

where

$$P(\eta) = \int_\eta^\infty \frac{(1 - e^{-t})^2}{t^2} dt \, .$$

If we write (7) as

$$L(u) = -\frac{\Gamma^2 U}{8\nu x}\left(P + \eta \frac{dP}{d\eta}\right) = h(\eta)$$

then a solution of (7) can be made up of linear combinations of solutions of $L(u) = 0$, plus a "particular" solution of $L(u) = h(\eta)$. Clearly, one solution of $L(u) = 0$ is $u = U$. If we anticipate $u = g(x)f(\eta)$, then substitution into $L(u) = 0$ readily yields two equations for g and f:

$$\begin{gathered} xg'(x) = cg(x) \\ \eta f''(\eta) + (\eta + 1)f'(\eta) - cf(\eta) = 0 \end{gathered} \tag{8}$$

where c is a numerical constant. The solutions for g and f which vanish as $x \to \infty$ and $\eta \to \infty$, and which remain finite at $\eta = 0$ are

$$g(x) = x^c \qquad c < 0$$

and (9)

$$f(\eta) = \Phi(-c, 1, -\eta)$$

where $\Phi(a, c, x)$ is the Confluent Hypergeometric function. The asymptotic behavior of $\Phi(-c, 1, -\eta)$ as $\eta \to \infty$ is [2, p. 278]

$$\Phi(-c, 1, -\eta) = \frac{1}{\Gamma(1+c)} \frac{1}{\eta^{-c}}\left[1 + O\left(\frac{1}{\eta}\right)\right] \qquad \eta \to \infty \, .$$

In particular, we note that $\Phi(-c, 1, -\eta)$ has no zero if $0 < c \leq 1$, and the number of zeros of $\Phi(-c, 1, -\eta)$ equals the smallest integer equal or greater than $(-c-1)$. See [2, p. 289].

Generalizing the above, we can seek $u = g_1(x)f_1(\eta) + g_2(x)f_2(\eta)$. Indeed, for $g_1 = \log x/x$ and $g_2 = x^{-1}$, we have the equations for f_1 and f_2 as

$$\begin{aligned} &\eta f_1'' + (\eta+1) f_1' + f_1 = 0 \\ &\eta f_2'' + (\eta+1) f_2' + f_2 = f_1 \end{aligned} \tag{10}$$

from which we recover part of the Batchelor solution

$$f_1 = -e^{-\eta} \quad \text{and} \quad f_2 = e^{-\eta} \int_0^\eta \frac{1-e^t}{t}\, dt\,.$$

One can go on to seek solutions to $L(u)=0$ in the form

$$\sum_{i=1}^{k} g_i(x) f_i(\eta)$$

but our point to illustrate that the Batchelor solution is not the only admissible one has been served and we shall not pursue this further. A "particular" solution to (7) is readily constructed as

$$u = \frac{\Gamma^2}{8\nu x} e^{-\eta} \int_0^\eta P(t) e^t dt \tag{11}$$

and an admissible solution is therefore the sum of (11) and any solution of $L(u)=0$, providing the sum total satisfy the imposed boundary conditions.

Clearly then, within the framework of the linearized system, it is not possible to single out an appropriate solution without further information or constraints. The next question then is: Can the situation be rectified if we proceed to consider a non-linear system? Here, we have a few choices. We can consider the non-linear version corresponding to the system (1) or (3), or a somewhat more complicated system used by Hall. However, the solution of the linear equations makes it clear that the coordinate system most suitable for the flow field under investigation is the parabolic coordinates. It therefore seems logical to write the Navier–Stokes equations in parabolic coordinates, and to seek simplifications that are valid far downstream.

3. Formulation in parabolic coordinates

Let (α, β) be general orthogonal coordinates in the meridian plane with scale factors h_1, h_2; and ϕ the azimuthal angle with scale factor $h_3 = r$. Then with (u, v, w) denoting the velocity components in the directions of increasing (α, β, ϕ), the equations of motion in terms of the Stokes stream-function ψ, the angular momentum $\Omega = h_3 w$ ($\Omega \equiv C$ of section 2), and the ring circulation density l, defined by $l = -D^2\psi/h_3^2$, are [3, p. 114]:

$$h_2 h_3 u = \frac{\partial \psi}{\partial \beta}\,; \quad h_1 h_3 v = -\frac{\partial \psi}{\partial \alpha}\,; \tag{12}$$

$$\frac{2\Omega}{h_3}\frac{\partial(\Omega, h_3)}{\partial(\alpha,\beta)} + \frac{\partial(\psi, h_3^2 l)}{\partial(\alpha,\beta)} - 2h_3 l\,\frac{\partial(\psi, h_3)}{\partial(\alpha,\beta)} = -\nu h_1 h_2 h_3 D^2(h_3^2 l)\,; \tag{13}$$

and

$$\frac{\partial(\psi, \Omega)}{\partial(\alpha,\beta)} = -\nu h_1 h_2 h_3 D^2 \Omega \tag{14}$$

where

$$D^2 = \frac{h_3}{h_1 h_2}\left[\frac{\partial}{\partial \alpha}\left(\frac{h_2}{h_1 h_3}\frac{\partial}{\partial \alpha}\right) + \frac{\partial}{\partial \beta}\left(\frac{h_1}{h_2 h_3}\frac{\partial}{\partial \beta}\right)\right].$$

We define parabolic coordinates (ζ, η) in the meridian plane by

$$x = \zeta - \eta\,; \quad r = 2(\zeta\eta)^{\frac{1}{2}} \tag{15}$$

where (x, r, ϕ) are cylindrical coordinates. The length element is given by

$$ds^2 = h_1^2 d\zeta^2 + h_2^2 d\eta^2 = \frac{\zeta+\eta}{\zeta} d\zeta^2 + \frac{\zeta+\eta}{\eta} d\eta^2\,.$$

The surface $\eta=k=$constant is a paraboloid of focal length k; and $\eta=0$ is the axis.

If we denote dimensional quantities by asterisks, we can non-dimensionalize the governing equations by using the axial velocity far upstream U, the kinematic viscosity ν, a characteristic length a, and the angular momentum at infinity Γ. We define

$$\zeta^* = \frac{\nu}{U}\zeta \; ; \quad \eta^* = \frac{\nu}{U}\eta \; ; \quad \psi^* = \frac{2\nu^2}{U}\psi$$

$$l^* = \frac{U^3}{2\nu^2}l \; ; \quad R = \frac{Ua}{\nu} \; ; \quad \Omega^* = \Gamma\Omega \, .$$

The dimensionless governing equations are

$$\zeta\frac{\partial^2\psi}{\partial\zeta^2} + \eta\frac{\partial^2\psi}{\partial\eta^2} = -\zeta\eta(\zeta+\eta)l \tag{16}$$

$$\zeta\frac{\partial^2 l}{\partial\zeta^2} + \eta\frac{\partial^2 l}{\partial\eta^2} + 2\frac{\partial l}{\partial\zeta} + 2\frac{\partial l}{\partial\eta} = -\frac{\partial(\psi, l)}{\partial(\zeta,\eta)} - T\frac{\Omega}{\zeta\eta}\frac{1}{\eta}\frac{\partial\Omega}{\partial\zeta} - \frac{1}{\zeta}\frac{\partial\Omega}{\partial\eta} \tag{17}$$

and

$$\zeta\frac{\partial^2\Omega}{\partial\zeta^2} + \eta\frac{\partial^2\Omega}{\partial\eta^2} = -\frac{\partial(\psi,\Omega)}{\partial(\zeta,\eta)} \tag{18}$$

The dimensionless group $T=\Gamma^2/(4\nu^2)$ is the Taylor number. We are interested in solutions of the above system valid far downstream. The appropriate boundary conditions to be imposed are:

$$\text{as } \eta\to\infty: \quad \Omega=1 \; ; \; \frac{\partial\psi}{\partial\eta} = \zeta \; ; \tag{19}$$

$$\text{at } \eta=0 \; ; \quad \Omega=\psi=0 \; ; \; \frac{\partial\psi}{\partial\eta} = \zeta - K_1 g_1(\zeta) \, . \tag{20}$$

Here, $\psi=\zeta\eta$ represents the uniform stream. The last condition at $\eta=0$ is to accommodate a velocity defect. The constant K_1 is a parameter of the problem, and $g_1(\zeta)$ is to be determined.

With the above information, we attempt to construct an asymptotic solution for $\zeta\gg\eta$, $\zeta\to\infty$, of the form

$$\begin{aligned} &\psi(\zeta,\eta) \sim \zeta\eta + g_1(\zeta)\psi_1(\eta) + g_2(\zeta)\psi_2(\eta) + \dots \\ &\Omega(\zeta,\eta) \sim \Omega_0(\eta) + f_1(\zeta)\Omega_1(\eta) + f_2(\zeta)\Omega_2(\eta) + \dots \end{aligned} \tag{21}$$

The asymptotic sequences $\{g_n(\zeta)\}$ and $\{f_n(\zeta)\}$ are to be determined as we proceed. In this note, we shall obtain only $g_1\psi_1$ and $f_1\Omega_1$. Higher order terms can be determined in an iterative manner and so will not be presented. Obviously, we require $g_1(\zeta)=o(\zeta)$, $f_1(\zeta)=o(1)$, and we assume

$$f_i^{(n)}(\zeta) = o(f_i^{(n-1)}(\zeta) \; ; \quad g_i^{(n)}(\zeta) = O g_i^{(n-1)}(\zeta) \, .$$

Substitution of (21) into (16), (17) and (18) yield, to the lowest order:

$$l = -\frac{g_1(\zeta)}{\zeta^2}\psi_1''(\eta) \tag{22}$$

$$g_1(\zeta)[\eta^2\psi^{(4)} + 2\eta\psi_1^{(3)} + \eta^2\psi^{(3)} + 2\eta\psi_1''] = T\Omega_0\Omega_0' \tag{23}$$

$$\Omega_0'' + \Omega_0' = 0$$

As the equation for ψ_1 depends on the choice of $g_1(\zeta)$, we must decide at this point how it should be chosen. For example, if we choose $g_1(\zeta)=\log\zeta$, or $g_1(\zeta)=\zeta^c$, $0<C<1$; which certainly satisfy $g_1(\zeta)=o(\zeta)$, then to the lowest order the equation would be

$$\eta^2\psi_1^{(4)} + 2\eta\psi_1^{(3)} + \eta^2\psi_1^{(3)} + 2\eta\psi_1'' = 0 \, .$$

If we choose $g_1(\zeta)=1$, then the ψ_1 equation becomes

$$\eta^2 \psi_1^{(4)} + 2\eta \psi_1^{(3)} + \eta^2 \psi_1^{(3)} + 2\eta \psi_1'' = T\Omega_0 \Omega_0' . \tag{24}$$

Now the problem at hand is one in which we want to examine the interaction of swirl with the uniform stream. If we choose $g_1(\zeta)$ so that the ψ_1 equation is independent of Ω_0, the interaction is relegated to a higher order term. Such a situation is contrary to our objective. Hence, it seems logical that we choose $g_1(\zeta)=1$, thereby getting (24) as the equation of ψ_1. Once this is done, we can readily obtain the solutions of Ω_0 and ψ_1 as

$$\begin{aligned} &\Omega_0 = 1 - e^{-\eta} \\ &\psi_1'(\eta) = \frac{T}{2} \int_0^\eta e^{-x} dx \int_0^x e^x P(t) dt - K_1 (1 - e^{-\eta}) \end{aligned} \tag{25}$$

where $P(\eta)$ is as defined in (7). Hence, by considering the equations of motion as parabolic coordinates, and by insisting that the interaction of the velocity components be a first order effect, we have resolved the difficulty encountered in the previous section.

We now turn to the Ω-equation to determine Ω_1. Again by substituting (21) in (18), we readily see that $f_1(\zeta)$ must be chosen as ζ^{-1} and the equation governing Ω_1 is

$$\eta \Omega_1'' + \eta \Omega_1' + \Omega_1 = 0 ,$$

implying that Ω_1 is not explicitly dependent on ψ_1. An appropriate solution is

$$\Omega_1(\eta) = C_1 \eta e^{-\eta} .$$

To determine the constant C_1, we proceed to construct the equation for ψ_2. We obtain in a direct manner $g_2(\zeta)=\zeta^{-1}$ and

$$\begin{aligned} \eta \psi_2^{(4)} + (\eta + 1)\psi_2^{(3)} + 2\psi_2'' &= -T \left[\frac{(\Omega_0 \Omega_1)'}{\eta} + \frac{\Omega_0 \Omega_1}{\eta^2} \right] - 2\psi_1' \psi_1'' \\ &= h(\eta) . \end{aligned}$$

From the above equation, we can produce an integral relation by integrating with respect to η from 0 to ∞; and by imposing the conditions that $\psi_2'(0)=0$ and $\psi_2'(\infty)$ vanishes sufficiently rapidly. This then yields the condition

$$\int_0^\infty h(\eta) d\eta = 0 ,$$

which gives the result

$$C_1 = \frac{K_1^2}{(2 \log 2) T} .$$

Hence, we have the following expansions for ψ and Ω:

$$\begin{aligned} &\psi \sim \zeta\eta - K_1(1 - e^{-\eta}) + \frac{T}{2} \int_0^\eta e^{-x} dx \int_0^x e^t P(t) dt + O\left(\frac{1}{\zeta}\right) \\ &\Omega \sim (1 - e^{-\eta}) + \frac{1}{\zeta} \frac{K_1^2}{(2 \log 2) T} \eta e^{-\eta} + O\left(\frac{1}{\zeta^2}\right) . \end{aligned} \tag{27}$$

Since $\zeta u \sim \partial\psi/\partial\eta$ for $\zeta \gg \eta$, all velocity components are given to $O(\zeta^{-2})$.

4. Concluding remarks

The solution we have obtained differs from Batchelor's solution for the linearized system in that there is no log x term in the expression for the axial velocity. Hence the axial velocity defect decays like x^{-1} instead of x^{-1} log x. Also, the axial velocity defect and the perturbation in the angular velocity are both functions of the Taylor number T. For a given K_1, the perturbational angular velocity is decreased as T increases, and the axial velocity defect becomes less appreciable except right at the axis. Indeed, as T increases beyond some critical value dependent on K_1, the axial velocity ψ' develops an overshoot, as is evident from (27).

Lastly, we mention that if K_1 is chosen to be a negative quantity, then the solution can be used to describe the motion of a swirling jet in a uniform stream. This problem arises from the consideration of flow over engine-wing combinations. The related problem of a swirling jet emerging into fluid at rest had been considered by Loitsianskii [6] and Görtler [4], and the solution given in terms of the similarly variable r/x.

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